MISCELLANEOUS EQUATIONS

Some miscellaneous equations can be solved by the ordinary methods, but some of them need special methods for their solutions. Here we are discussing some of them.

(A) Indeterminate Linear Equations: when we have only one equation involving two or more unknown variables, then there are infinite solutions, but with some restrictions on the variables, the number of solutions becomes limited. It is a known fact that if number of unknown quantities is greater than the number of independent equations, then there will be many solutions and the equations become indeterminate. But if the variables are positive integers, then solution to the equation becomes easier. Consider the following example.

Suppose there is only one equation $3x + 4y = 88$, where x and y are positive integers.

$\Rightarrow$ $4y = 88 - 3x$

Left hand side of the equation is a multiple of 4, hence right hand side is also a multiple of 4.

Therefore $3x$ must be a multiple of 4, hence minimum value of x is 4. Corresponding value of y is 19.

Now there will be more solutions other than (4, 19), in the solution set if x increases by 4 then y decreases by 3 or vice versa; in general x increases /decreases at the rate of coefficient of y and y decreases/increases at the rate of coefficient of x, where coefficients of x and y are co-prime to each other.

Now the complete solution set is

x	y
4	19
8	16
12	13
16	10
20	7
24	4
28	1

Hence total number of solutions is 7.

Note: It should be noted that series of values of x and y in the solutions of $ax \pm by = c$, form two Arithmetical progressions in which the common differences are b and a respectively, where a, b, c are positive integers and the equation $ax \pm by = c$ is in the simplest form.

***Example 1**: Solve $3x + 5y = 38$, where x and y are positive integers.*

Solution: $3x = 38 - 5y$

$\Rightarrow$ $3x = 33 - 6y + (5 + y)$. Since x and y are positive integers, hence $33 - 6y + (5 + y)$ must be divisible by 3, that means $(5 + y)$ is multiple of 3.

Hence minimum value of y is 1 and corresponding value of x is 11.

In order to get complete solution, values of x will change at the rate of 5, and values of y will change at the rate of 3. Since $3x + 5y$ is a constant, thus changes in x and y will be of opposite signs.

x	y
11	1
6	4
1	7

There will be 3 solutions.

***Example 2**: Solve the equation for positive integers $17x - 15y = 50$.*

Solution: $15y = 17x - 50$

$\Rightarrow 15y = 15x - 45 + 2x - 5$

$\Rightarrow$ $2x - 5$ is a multiple of 15, so $x = 10$ and corresponding value of y is 8. This general solution will be $x = 15k + 10$, $y = 17k + 8$.

Number of solutions will be infinite. These solutions are obtained by assigning to k the values 0, 1, 2, ……

***Example 3**: Divide 81 into two parts so that one is multiple of 5 and other is a multiple of 8.*

Solution: $5x + 8y = 81$

$\Rightarrow$ $5x = 81 - 8y = 80 - 10y + 2y + 1$. Now $2y + 1$ is a multiple of 5, hence $y = 2$ and corresponding value of x will be 13.

x	y
13	2
5	7

Hence the two parts are (65, 16) or (25, 56).

***Example 4**: Find the smallest value of x, if $17x = 120y + 1$, where x and y are natural numbers.*

Solution: Since x and y are both natural numbers. $17x$ is multiple of 17, hence $120y + 1$ should also be multiple of 17.

$17x = 120y + 1$ can be written as:

$17x = 119y + \underbrace{y+1}$, where $(y + 1)$ must be a multiple of 17. We can put $y + 1$ as 17 or $y = 16$.

By putting $y = 16$, we get $x = 113$

(B) Solution to the equation of the type $xy = ax + by$, where x, y, a and b are integers can obtained by reorganizing the equation as:

$xy - ax - by = 0 \Rightarrow xy - ax - by + ab = ab$
$\Rightarrow (x - b)(y - a) = ab$

Since $(x - a)$ and $(y - b)$ are both integers, hence number of solutions can be obtained by calculating factors of ab.

Example 5: *If x and y are positive integers, find the number of solutions of $xy = x + y + 25$.*

Solution: $xy = x + y + 25$
$\Rightarrow xy - x - y + 1 = 25 + 1 \Rightarrow (x - 1)(y - 1) = 26$

Since x and y are positive integers hence, $(x - 1)$ can take 4 values i.e. 1, 2, 13 and 26. Therefore x can be 2, 3, 14 and 27. Corresponding values of y are 27, 14, 3 and 2. Note that here 1, 2, 13 and 26 are factors of 26. Hence there are 4 solutions to the given equation.

Example 6: *If x and y are integers then find the number of solutions of $xy = 2x + 3y + 25$.*

Solution: $xy - 2x - 3y = 25$
$\Rightarrow xy - 2x - 3y + 6 = 25 + 6 \Rightarrow (x - 3)(y - 2) = 31$

Here $(x - 3)$ can be 1, 31, –1, – 31. Hence there are 4 solutions including negative values.

Example 7: *If $xy + 3x + 2y = 24$ and x, y belongs to set of natural numbers then how many solutions sets are possible for (x, y)?*

Solution: Given that $xy + 3x + 2y = 24$

$\Rightarrow xy + 3x + 2y + 6 = 30$

$\Rightarrow (x + 2)(y + 3) = 30$

$x + 2$	$y + 3$
1	30
2	15
3	10
5	6
6	5
10	3
15	2
30	1

Since x and y are natural numbers, hence $x + 2$ and $y + 3$ are more than 2 and 3 respectively. So $x + 2$ can be 3, 5, 6 only and corresponding values of $y + 3$ are 10, 6 and 5.

There are three possible solution sets.

(C) Solution to the equation $x^2 - y^2 = n$, where x, y and n are integers, can be obtained by calculating factors of n, as the equation can be written as $(x + y)(x - y) = n$

Example 8: *Find the number of solutions of the equation $x^2 - y^2 = 31$, where x and y are integers.*

Solution: The given equation $x^2 - y^2 = 31$.
$\Rightarrow (x + y)(x - y) = 31$
Here $(x + y)$ can be 1, 31, – 1 and – 31 and corresponding values of $(x - y)$ can be 31, 1, – 31 and – 1. Therefore value of (x, y) can be (16, – 15), (16, 15), (– 16, 15), (– 16, – 15).

Example 9: *If x and y are natural numbers, then find the number of solutions of $x^2 - y^2 = 45$.*

Solution: $x^2 - y^2 = 45$
$\Rightarrow (x + y)(x - y) = 45$. Note that for natural values of x and y, $(x + y) > (x - y) \Rightarrow (x + y)$ can be 45, 15 and 9. Corresponding values of $(x - y)$ can be 1, 3 and 5.

Solutions are (23, 22), (9, 6) (7, 2).

(D) Solution of the equation $x^2 + y^2 = 0$, where x and y are real numbers can be done using the fact that square of any real number is always non-negative and sum of two or more squares is zero if and only if all the squares are zero. Consider the following example.

Example 10: *Find the number of solutions of the equation $x^2 - 2x + y^2 + 1 = 0$.*

Solution: The given equation is $(x - 1)^2 + y^2 = 0$. Since $(x - 1)^2$ and y^2 are both non negative and their sum is zero, hence $x = 1$, $y = 0$. There is only one solution.

Example 11: *Find the number of solutions of the equation $x^2 - 2xy + 2y^2 - 4y + 4 = 0$.*

Solution: The given equation can be written as $(x - y)^2 + (y - 2)^2 = 0$
$\Rightarrow x = y = 2$. Hence there is only one solution.

(E) Solution of the equation $x^2 + y^2 = n$ can be done using some divisibility rules which are discussed below.

1. No square ends with 2, 3, 7 or 8.
2. Every square is either multiple of 3 or 1 more than multiple of 3.
3. Every square is either multiple of 4 or 1 more than multiple of 4.

Example 12: *Find the number of solutions of the equation $x^2 + y^2 = 1003$.*

Solution: We know that every square is either multiple of 4 or 1 more than multiple of 4. So $x^2 + y^2$ can not be of the form of $4k + 3$. Since 1003 is of the form of $4k + 3$, hence there is no solution.

Example 13: *Find the number of solutions of the equation $x^2 + y^2 = 290$.*

Solution: Since 290 is of the form of $3k + 2$, hence each of x^2 and y^2 is of the form of $3k + 1$. Also both x and y are odd.

x is less than 18, odd number and not a multiple of 3, hence x can be 1, 5, 7, 11, 13, 17.

x	y
1	289
11	169
13	121
17	1

MISC EXAMPLES:

Example 14: *The prices of a pen, pencil and eraser, are Rs. 5, Rs 1 and 5 paise respectively. In how many ways 100 items, can be purchased in Rs. 100 such that at least one pen is purchased.*

Solution: Suppose number of pens, pencils and erasers purchased are x, y and z respectively, then

$x + y + z = 100$ (i)

$5x + y + \frac{z}{20} = 100$

Or $100x + 20y + z = 2000$ (ii)

From equation (ii) – equation (i)

$99x + 19y = 1900$

In the solution x and y will change at the rate of 19 and 99 respectively. Initial values of (x, y) are (0, 100). Next solution will be (19, 1). Hence $(x, y, z) = (19, 1, 80)$ and there is only one solution.

Example 15: *A student purchased some pens, pencils and erasers; the prices of one pen one pencil and one eraser are Rs. 5, 1 and 0.5 per piece respectively. He had a total of Rs.100 and he purchased a total of 100 items. If he purchased more number of pens than pencils, find the number of erasers, he purchased.*

Solution: Suppose number of erasers, pencils and pens are a, b and c then

$a + b + c = 100$ ...(1)

$\frac{a}{2} + b + 5c = 100$ ...(2)

2× equation (2) – equation (1), $b + 9c = 100$, now

b	c
1	11
10	10
19	9
28	8

By the above table we see that there is only one case where $c > b$ hence number of pencils = 1, number of pens = 11, Hence number of erasers =100 – (1 + 11) = 88

Example 16*: 11x +15y = 1031, where x, y belong to natural numbers*

Solution: $11x = 1031 – 15y$

Since left hand side of the equation is multiple of 11, hence right hand side is also multiple of 11. So the remainder when right hand side is divided by 11 must be either zero or multiple of 11.

$11x = 1022 - 11y + \underbrace{8 - 4y}$

Hence $8 – 4y = 0$ or $y = 2$ and $x = 91$

Example 17: *A student bought a dozen pieces of pencils and pens for Rs 132. If a pen costs Rs 3 more than a pencil and more pens are purchased than pencils were purchased, how many of each pen and pencils were bought?*

Solution: let x be the number of pens, then $(12 – x)$ is the number of pencils. Also suppose y is the price of one pencil, then price of one pen will be $(y + 3)$.

$(y + 3)x + (12 – x)y = 132 \Rightarrow 3x + 12y = 132$

$\Rightarrow x + 4y = 44$.

Now in this equation, $4y$ and 44 both are multiple of 4, hence x must be a multiple of 4. When x is 4, y is 10. Also in the solutions x will increase at the rate of 4 and y will decrease at the rate of 1. The following table gives the values of x and y.

x	y
4	10
8	9
12	8
...	...

As we know that total number of pens and pencils is 12, hence value of x cannot be more than or equal to 12. Also number of pens is more than number of pencils. Thus the number of pens is 8 and number of pencils is 4. Price of one pen is Rs 12, and price of one pencil is Rs 9.

Example 18: *Four friends Anil, Monica, Vinod and Vinay buy some pens, pencils and erasers from a shop. Anil buys 6 pens, 5 pencils and 1 eraser for Rs. 47.50. Monica buys 3 pens, 3 pencils and 2 erasers for Rs. 27. Vinod buys 5 pens, 2 pencils and 5 erasers for Rs 41.50. How much will it cost Vinay if she buys 3 pens, 2 pencils and 5 erasers?*

Solution: Let the price of each pen, pencil and eraser be x, y and z respectively. Therefore we get,

$6x + 5y + z = 47.50$ ...(1)

$3x + 3y + 2z = 27$ ...(2)

$5x + 2y + 5z = 41.50$ ...(3)

Let us first eliminate one variable. By adding (2) and (3) and subtracting (1) from this (to eliminate y) we get,

$2x + 6z = 21$ ...(4)

Eliminating y from (2) and (3) by multiplying (2) by 2 and (3) by 3 and subtracting we get, $9x + 11z = 70.5$ ...(5)

Multiplying (5) by 2 and (4) by 9 and subtracting we get, $32z = 48$ or z = Rs. 1.50 and $x = 6$, and $y = 2$. The cost of one pen is Rs. 6, the cost of 1 pencil is Rs. 2 and that of eraser is Rs. 1.50. Therefore, the cost of 3 pens, 2 pencils and 5 erasers is $(3\times6 + 2\times2 + 5\times1.5)$ = Rs. 29.50.

Example 19: *Newton purchased certain apples of type A and a certain number of apples of type B. Prices of apples of type A & B are Rs. 7 and Rs. 5 respectively if he spent a total of Rs. 114. In how many ways he can purchase the apples?*

Solution: Suppose he purchased a apples of type A and b apples of type B then $7a + 5b = 114$. By simple observation if we put $a = 2$, we get $b = 20$

a	b
2	20
7	13
12	6

As A increases from its lowest value of 2 at the rate of 5 and B decreases form its highest value of 20 at the rate of 7, hence total number of solutions = 3.

1. Anil bought a total of 30 white and black pencils for a total of Rs. 32. The cost of each white pencil is 70 paise more than the cost of each black pencil. Which of the following represents a possible value of the cost of each black pencil (in paise)?
(1) 50 (2) 46
(3) 35 (4) 40
(5) None of these

2. Reena has a certain amount of money in only 2 Rupee and 20 Rupee notes. The number of 2 Rupee notes multiplied by number of 20 Rupee notes is equal to one fourth of total money that she has. Then which of the following is always false?
(1) Total number of notes with Reena is a prime number
(2) Total number of notes with Reena is a perfect square
(3) Total number of notes with Reena is more than 10
(4) Total number of notes with Reena is less than 10
(5) None of these

3. A man had x coins in the beginning. He went for gambling with x coins and played 15 rounds. In each round he doubled his coins. After each round of gambling, he gave y coins to his friend. At the end of 15^{th} round after giving y coins to his friend he was left with 32767 coins then how many of the following statements are true.
(1) Least value of y is 32767
(2) Least value of x is 32767
(3) If y is 327679, then x will be 327670
(4) If x is 98301, then y will be 98303
(1) 0 (2) 1
(3) 2 (4) 3
(5) 4

4. Find the number of solutions in natural number for x and y for $11x + 15y = 1031$
(1) 5 (2) 6
(3) 7 (4) 8
(5) None of these

5. If $775x - 711y = 1$ and x, y are natural numbers, find the least value of $x + y$.
(1) 209 (2) 208
(3) 207 (4) Can't say
(5) None of these

6. In how many ways Rs. 500 can be paid with the help of 50 Rs. notes and 5 Rs. notes, including the cases when he can pay Rs. 500 with only one type of notes
(1) 10 (2) 11
(3) 12 (4) 13
(5) None of these

7. The expenses of a party having 43 people was 229 Rs. If each man paid 10 Rs. each women paid 5 Rs. and each child paid 2 Rs. How many solutions are possible?
(1) 4 (2) 5
(3) 3 (4) 2
(5) None of these

8. Find the product of x, y, z and u, given that x, y, z, u are real and $4x^2 + 9y^2 + 16z^2 + u^2 + 4x - 12y - 24z + 8u + 30 = 0$
(1) 1 (2) –1
(3) $\frac{1}{4}y$ (4) data insufficient
(5) None of these

9. $x^2 + 8y^2 + 18z^2 + 16u^2 - 4xy - 12yz - 24zu = 0$ and x, y, z, u are real find the value of $\frac{x}{y}+\frac{y}{z}+\frac{z}{u}+\frac{u}{x}$
(1) $5\frac{1}{12}$ (2) 5
(3) $4\frac{11}{12}$ (4) data insufficient
(5) None of these

10. A two digit number is $\frac{8}{3}$ times of the product of its digits, how many such numbers are possible?
(1) 2 (2) 1
(3) 0 (4) 3
(5) 4

11. If P, Q and R are distinct single digit numbers. If PQ and QR denote the two digit numbers formed with the respective digits and $\frac{PQ}{P+Q}=\frac{QR}{Q+R}$, how many values of (P, Q, R) exist?
(1) 8 (2) 6
(3) 12 (4) 3
(5) 4

12. If each pen costs Rs.37 and each pencil costs Rs.23. I bought some pens (m pens) and some pencils (n pencils) and my total expenses were 752 Rs. Find least value of mn.

(1) 165 (2) 154
(3) 180 (4) 150
(5) data insufficient

13. If $x + y + z = 40$ and $4x + 2y + 17z = 310$ and x, y, z are natural numbers. How many solution set for (x, y, z) are there?

(1) 2 (2) 1
(3) 3 (4) data insufficient
(5) None of these

14. If $4x^2 - y^2 - 315 = 0$ and x, y are natural numbers. How many solution set for (x, y) exist?

(1) 5 (2) 6
(3) 7 (4) data insufficient
(5) None of these

15. In how many ways "12345678906" can be written as difference of two perfect squares

(1) 0 (2) 8
(3) 64 (4) 16
(5) None of these

16. If $xy - 9x - 7y = 57$ and x, y are natural numbers. How many solutions are possible for (x, y)

(1) 14 (2) 18
(3) 16 (4) 20
(5) None of these

17. If $2xy - 4x + y = 17$; $3yz + y - 6z = 52$; $6xz + 3z + 2x = 29$. How many solutions exists for (x, y, z)

(1) 1 (2) 2
(3) 3 (4) 4
(5) None of these

18. If m and n are two integers and $m^2 + n^2 = 37$, find the number of solutions

(1) 8 (2) 4
(3) 2 (4) 16
(5) 32

19. If $\frac{4-2y}{|x|} = \frac{9+x}{3|y|} = 1$, which of the following could be value of (x, y)

(1) (–6, –1) (2) (6, 5)
(3) (–2, 1) (4) (–2, –1)
(5) None of these

20. How many two digit numbers does not follow, product of digits is greater than sum of digits?

(1) 27 (2) 63
(3) 26 (4) 28
(5) None of these

21. A man presents a cheque of an amount in denomination of rupees and paise, to a bank. But the cashier of the bank misreads the amount written on the cheque and pays him the rupee amount in place of paise and vice versa. The man spends 20 paise and finds that he is left with double of the original amount written on the cheque. If the initial amount is p, then which of the following is correct?

(1) Rs 26 < p < Rs 26.45
(2) Rs 26.6 < p < Rs 26.95
(3) Rs 37.45 < p < Rs 37.77
(4) Rs 37.60 < p < Rs 38.50
(5) None of these

22. If $119x + 238y + 7z = 2142$, where x, y, z are natural numbers, then find the value of z

(1) 117 (2) 234
(3) 351 (4) 468
(5) None of these

23. A test has 100 questions. Each correct answer is awarded (+2) marks; each wrong answer is awarded (–1.5) mark and each unanswered question is awarded (– 0.75) marks. Now how many of the following statements are true? if the number of questions which are correct, wrong and unanswered are x, y, z respectively

(1) If a candidate has scored 5 marks, maximum value of x, y, z can be 43, 51, 62 respectively

(2) If a candidate has scored 5 marks, minimum value of x, y, z can be 31, 7, 6 respectively

(3) If a candidate has scored 5 marks, then 5 solution sets exist for (x, y, z)

(4) If x, y, z are divided by 3, 11, 14 respectively we get remainders as 1, 7, 6 respectively and the score of the candidate was 5

(1) 0 (2) 1
(3) 2 (4) 3
(5) 4

24. How many two digits number follow the rule $P + S = N$, where P is the product of digits, S is the sum of digit and N is the two digit number.

(1) 9 (2) 10
(3) 18 (4) 20
(5) None of these

25. How many three digit numbers follow the rule: $1.5P + S = N$, where P = product of the digits of the number, S = sum of the digits of the number, N is the three digit number.

(1) 2 (2) 3
(3) 1 (4) 0
(5) None of these

26. Find the number of solutions of the equation $3x + 4y = 1$, where x and y are integers and $-50 \le x \le 50$, $-60 \le y \le 60$.

(1) 24 (2) 25
(3) 49 (4) 50
(5) None of these

27. If x and y are integers, find the number of solutions of the equation $x^2 + y^2 = 3000$

(1) 4 (2) 8
(3) 2 (4) 1
(5) 0

28. Find the number of solutions of the equation $x^2 - 2x + y^2 = 25$, where x and y are integers.

(1) 2 (2) 4
(3) 6 (4) 8
(5) None of these

29. If n is an integer and $n^2 + 10n + 14$ is a perfect square, then how many values of n are possible?

(1) 1 (2) 2
(3) 3 (4) 3
(5) 0

30. If ℓ, m and n are non negative integers, find the number of solutions of the equation $\ell mn + \ell m + mn + \ell n + \ell + m + n = 100$

(1) 1 (2) 2
(3) 3 (4) 4
(5) 0

ANSWER KEY

1.	(2)	**11.**	(1)	**21.**	(5)
2.	(5)	**12.**	(1)	**22.**	(5)
3.	(5)	**13.**	(1)	**23.**	(5)
4.	(3)	**14.**	(2)	**24.**	(1)
5.	(1)	**15.**	(1)	**25.**	(1)
6.	(2)	**16.**	(3)	**26.**	(2)
7.	(2)	**17.**	(2)	**27.**	(5)
8.	(1)	**18.**	(1)	**28.**	(4)
9.	(1)	**19.**	(1)	**29.**	(2)
10.	(1)	**20.**	(1)	**30.**	(3)