INTRODUCTION

How important is to learn formulas:

Formulas are important in mathematics because they provide a concise and precise way to express mathematical relationships and operations. Learning formulas can certainly increase performance in mathematics, as formulas provide a way to quickly and accurately solve problems, and understand mathematical concepts.

Keep in mind that learning formulas is one of the initial stages of mastering mathematics, students will need many more skills to ace mathematics other than just learning the formulas. In higher level mathematical questions, students may need to have a strong foundation in mathematical proofs, logic, and abstract thinking before they can fully understand and apply formulas.

How to improve Mathematical Skills:

Here are a few ways to improve mathematics:

- ✓ **Practice, practice, practice:** Solving a variety of mathematical problems is the best way to improve your understanding and proficiency in math.
- ✓ **Understand the concepts:** Try to understand the underlying concepts of mathematical problems, not just the procedures for solving them.
- ✓ **Use visual aids:** Diagrams, graphs, and other visual aids can help you understand mathematical concepts and problem-solving strategies.
- ✓ **Learn by doing:** Try to apply mathematical concepts to real-world situations. This will help you see the relevance and practicality of what you're learning.
- ✓ **Engage with the material:** Ask questions, participate in class discussions, and take an active interest in your math education.
- ✓ **Be patient:** Improving in mathematics can take time and effort. Don't get discouraged if you don't see progress immediately.
- ✓ **Keep a positive attitude:** Stay motivated and believe in yourself. Remember that everyone struggles with math at times, and that it's normal to make mistakes.

01 Algebraic Identities & Vedic Mathematics

1. $a^2 - b^2 = (a+b)(a-b)$

2. $(a \pm b)^2 = a^2 + b^2 \pm 2ab$

3. $\left(a \pm \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} \pm 2$

4. $(a \pm b)^3 = a^3 \pm b^3 \pm 3ab(a \pm b)$

5. $a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$

6. $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$

7. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

8. $a^3 + b^3 + c^3 - 3abc$
$= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

9. $a^2 + b^2 + c^2 - ab - bc - ca$ is always more than or equal to zero as it is equal to $\frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2}$

10. If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

and $a^4 + b^4 + c^4 = \frac{(a^2 + b^2 + c^2)^2}{2}$

11. $(a+b+c)^3 = a^3 + b^3 + c^3 + 3ab(a+b) + 3bc(b+c) + 3ca(c+a) + 6abc$

12. If $a + \frac{1}{a} = 2$, then $a = 1$ only and if $a + \frac{1}{a} = -2$, then $a = -1$ only.

VEDIC MATHS

Vedic Mathematics is an ancient Indian system of mathematics which includes a set of techniques and shortcuts to perform mathematical calculations quickly and easily. Here is a Vedic Maths shortcut to calculate squares:

Base Method to calculate square:

This method is very helpful to calculate squares of the numbers which are close to 100. Here are the steps to calculate the square of a number that is close to 100.

Step 1: Find the difference between the number you want to square and 100. For example, if you want to find the square of 103, the difference is 3.

Step 2: Square the difference obtained in step 1 and represent this using 2 digits. In this case, 3×3 = 09. If the square contains more than 2 digits, then a carry is generated that is added in step 3.

Step 3: **Add** the difference to the number you want to square. In this case, 103 + 03 = 106.

Step 4: Append the square of the difference obtained in step 4 to the sum obtained in step 3. Thus, the square of 103 is $\underset{\text{step 4}}{106}\ \underset{\text{step 3}}{09}$ = 10609.

Take another example: Let the number is 113.

Step 1: Find the difference between the number you want to square and 100. For example, if you want to find the square of 113, the difference is 13.

Step 2: Square the difference obtained in step 1 and represent this using 2 digits. In this case, 13×13 = 169. Here the last two digits are 69 and the carry 1 is generated.

Step 3: Add the difference to the number you want to square along with carry forward. In this case, $113 + 13 + \underset{\text{carry}}{1} = 127$

Step 4: Append the square of the difference obtained in step 4 to the sum obtained in step 3.

In this case, the square of 113 is $\underset{\text{step 4}}{127}\ \underset{\text{step 3}}{69}$ = 10609.

Same steps can be repeated when the number is less than 100. Suppose the number is 87.

Step 1: Find the difference between the number you want to square and 100. For example, if you want to find the square of 87, the difference is 13.

Step 2: Square the difference obtained in step 1 and represent this using 2 digits. In this case, 13×13 = 169. Here the last two digits are 69 and the carry 1 is generated.

Step 3: **Subtract** the difference to the number you want to square and add the carry forward. In this case, $87-13+\underset{\text{carry}}{1}=75$

Step 4: Append the square of the difference obtained in step 4 to the sum obtained in step 3. In this case, the square of 87 is $\underset{\text{step 4}}{75}\ \underset{\text{step 3}}{69}$ = 7569

$$(100+x)^2 = \underbrace{100+x+x+c}_{\text{Number+difference+carry}} \Big| \underset{\text{2 digits}}{x^2}$$

$$(100-x)^2 = \underbrace{100-x-x+c}_{\text{Number−difference+carry}} \Big| \underset{\text{2 digits}}{x^2}$$

This Base method can be extended to any power of 10. For example, if we want to calculate squares of the numbers which are closer to 1000, then we must take last three digits of the square of the difference of the number from 1000.

For example, let the number is 1012, the difference of this number from 1000 is 12. Square of the number 12 is 144.

Append 144 to the sum obtained of 1012 and 12. In this case, the square of 1012 is $\underbrace{1012+12}\ \underbrace{144}_{\text{last 3 digits}}$ = 1024144.

$$(1000+x)^2 = \underbrace{1000+x+x+c}_{\text{Number+difference+carry}} \Big| \underset{\text{3 digits}}{x^2}$$

$$(1000-x)^2 = \underbrace{1000-x-x+c}_{\text{Number−difference+carry}} \Big| \underset{\text{3 digits}}{x^2}$$

02 Linear Equations

Linear equations:

- ✓ The expression on the left of the equality sign is known as the left-hand side and the expression on the right side of the equality sign is known as the right-hand side.
- ✓ The same number can be added to both sides of the equation.
- ✓ The same number can be subtracted from both sides of the equation.
- ✓ Both sides of an equation can be multiplied or divided by the same **non-zero** number.
- ✓ Any term of the equation may be taken to other side with the sign changed this method is known as **transposition.**
- ✓ If $\frac{ax+b}{cx+d}=\frac{m}{n}$, then $n(ax+b)=n(cx+d)$,this method is known as **cross multiplication**.

General solution of two linear equations:

Two linear equations can be solved using substitution or elimination of the variables.

If the two equations are $a_1x+b_1y+c_1=0$ and

$a_2x+b_2y+c_2=0$, then

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

Application of Linear Equations:

There are many questions that require use of linear equations to solve them, some of them are listed here:

- ✓ Question based on numbers usually require linear equations. Any two-digit number xy, can be written as $10x+y$ and any three-digit number xyz, can be written as $100x+10y+z$.
- ✓ Many puzzles can be solved using linear equations.
- ✓ Questions based on ages can be easily solved using equations.

NATURE OF SOLUTIONS:

Suppose the two equations are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then if

(1) $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$, there is only one solution

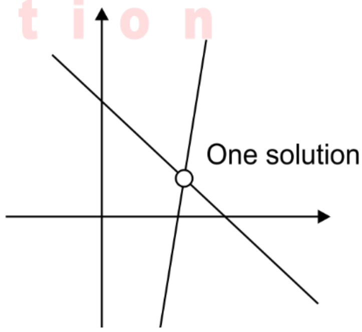
(2) $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$, there are many solutions

(3) $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$, there is no solution

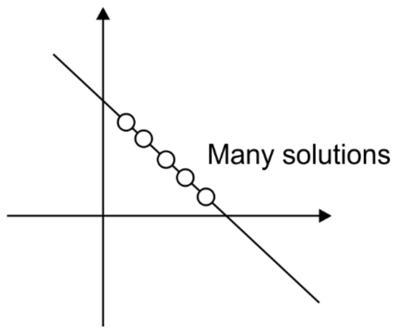
Graphical interpretation of the number of solutions of two linear equations:

Every linear equation represents a straight line when it is drawn on Cartesian system. **Geometrically every solution of an equation is a point on the line representing it.** Similarly, every solution of two equations is the common point of the lines representing the equations. There can be three possible ways in which two lines may intersect.

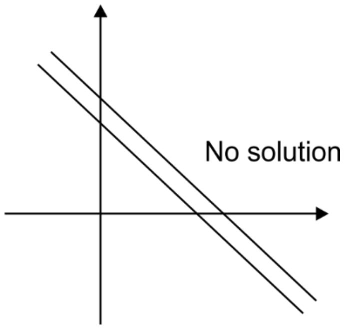
(a) If two equations are independent and there is only one solution, then these equations represent two straight lines which intersect at only one point and there is a unique solution.



(b) If two equations are dependent (that means that there are many solutions), then their graph is pair of lines coincident on each other.



(c) If two equations are inconsistent with no solution, then their graph is set of two lines parallel to each other



Indeterminate Equations

Any equation of the type $ax + by = c$ where a, b, c are positive integers, then this is known as an indeterminate equation and it can have an unlimited number of solutions.

When x, y are natural numbers then the solutions may be limited.

Consider the equation $4x + 7y = 74$, where x and y are positive integers.

By observation, we see that if $x = 1$, then $y = 10$. The next solution is $x = 8, y = 6$. Here x increases at a constant difference of 7 and y decreases at a constant difference of 4. We can generalize this as:

In the solutions of $ax + by = c$, where a, b, c, x, y are natural numbers, the values of x form a series that has a constant difference of b and the values of y also form a series that has a constant difference of a.

Example: If x and y are natural numbers, find the number of solutions of $4x + 3y = 103$.

Solution: The first solution can be found as $y = 1, x = 25$. In the subsequent solutions, the series formed by the values of x and y have constant differences. Have a look:

x	25	22	19	16	13	10	7	4	1
y	1	5	9	13	17	21	25	29	33

The values of x are decreasing at the rate of 3 and the values of y are increasing at the rate of 4. The total number of solutions = 9.

The indeterminate equations are quite useful in solving questions based on puzzles and numbers.

Number of Solutions of $ax + by = c$

The approximate number of solutions of the equation $ax + by = c$, where a, b, c are natural numbers and $\text{HCF}(a, b, c) = 1$ is given by

$$\frac{c}{\text{LCM}(a, b)}$$

For example, the number of solutions of $3x + 4y = 88$, where x and y are natural numbers is approximate to $\frac{88}{\text{LCM}(3, 4)} = 7$.

Note that this formula gives an approximate value of the number of solutions.

03 Ratio Proportion Variations

PROPERTIES OF RATIOS:

- ✓ If $\frac{a}{b}=\frac{c}{d}$, then $ad = bc$ and $\frac{a}{c}=\frac{b}{d}$ (Alternando)
- ✓ If $\frac{a}{b}=\frac{c}{d} \Rightarrow \frac{a-b}{a+b}=\frac{c-d}{c+d}$ (Componendo and dividendo)
- ✓ If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$, then k is also equal to:

  $$\frac{a+c+e}{b+d+f}=\frac{a+c}{b+d}=\frac{a-c-e}{b-d-f}=\frac{\sqrt{a^2+b^2+c^2}}{\sqrt{b^2+d^2+f^2}}$$

- ✓ Duplicate ratio of $\frac{a}{b}$ is $\frac{a^2}{b^2}$ and triplicate ratio of $\frac{a}{b}$ is $\frac{a^3}{b^3}$
- ✓ Sub duplicate ratio of $\frac{a}{b}$ is $\frac{\sqrt{a}}{\sqrt{b}}$ and sub triplicate ratio of $\frac{a}{b}$ is $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$.
- ✓ A ratio remains unchanged if its numerator and denominator are multiplied or divided by the same number.

Proportion:

It is an expression that shows that two ratios are equal and it is represented by '::'

Suppose two ratios $a:b$ and $c:d$ are equal, then

$a:b::c:d$

The numbers a, b, c, d are said to be in proportion.

Product of extremes = Product of means

$$ad = bc$$

Variation:

(a) Direct Variation: Two quantities x and y are said to be in direct variation, when their ratio is constant. In other words, when x increases, y also increases in the same ratio.

$\frac{x}{y}=\text{constant}=k$

$\Rightarrow \frac{x_1}{y_1}=\frac{x_2}{y_2}$, where x_1, y_1 and x_2, y_2 are the initial and final values of x and y.

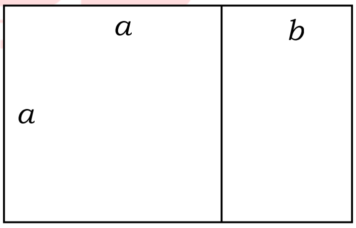
(b) Inverse Variation: Two quantities x and y are said to be in inverse variation, when their product is constant. In other words, when x increases, y decreases in such a way that

$xy=\text{constant}=k$

$\Rightarrow x_1y_1=x_2y_2$, where x_1, y_1 and x_2, y_2 are the initial and final values of x and y.

GOLDEN RATIO:

Golden ratio is the ratio of the length to the width of the most aesthetically pleasing rectangular shapes. This rectangle, called the **Golden Rectangle**, appears in nature and is used by humans in both art and architecture. The **Golden Ratio** can be noticed in the way trees grow, in the proportions of both human and animal bodies, and in the frequency of rabbit births.



This diagram shows a golden rectangle. The rectangle is divided into a square and a smaller rectangle. In a golden rectangle, the smaller rectangle is the same shape as the larger rectangle, in other words, their sides are proportional. In further words, the two rectangles are similar. This can be used as the definition of a golden rectangle. The proportions give us:

$$\frac{a}{b}=\frac{a+b}{a}$$

This fraction, $\left(\frac{a+b}{a}\right)$, is called the golden ratio (or golden section or golden mean).

The ratio is $\frac{\sqrt{5}+1}{2}:1$ **(close to 1.618).**

04 Percentage Profit and Loss

PERCENTAGE

$x\%$ of $y = \frac{xy}{100}$

PERCENTAGE CHANGE:

When a number is increased or decreased, then percentage change in the number is defined as

$$\% \text{ change} = \left[\frac{\text{final value} - \text{initial value}}{\text{intial value}}\right] \times 100.$$

SUCCESSIVE INCREMENT

Suppose a number is increased by $x\%$ then by $y\%$ if the initial value of the number is n, final value

$$= n\left(1+\frac{x}{100}\right)\times\left(1+\frac{y}{100}\right)$$

Alternatively, we can use the successive change formula,

$$\text{Effective percentage increase } = \left(x+y+\frac{xy}{100}\right)\%$$

For more than 2 successive changes, we can group them and apply the formula given above, for example if a number is successively increased by 10%, 20% and 30%, then we can first combine 10% and 20%, and the equivalent change is $10+20+\frac{10\times 20}{100} = 32\%$

Now equivalent change for 32% and 30% can be calculated as $32+30+\frac{32\times 30}{2} = 71.6\%$

If a quantity changes by three consecutive changes of $x\%$, $y\%$ and $z\%$, then the effective percentage change

$$= \left[x+y+z+\frac{xy}{100}+\frac{yz}{100}+\frac{zx}{100}+\frac{xyz}{10000}\right]$$

Note that for successive decrement, we can just use – sign and apply the same formula. For example, a number is first increased by 12% and then decreased by 11%, the effective percentage change = $12-11+\frac{(12)(-11)}{100} = -0.32\%$

Profit & loss:

Profit and loss are determined by the value of cost price and selling price cost price is the price at which an article is purchased and selling price is the price at which article is sold

Profit = selling price – cost price

Loss = cost price – selling price

$$\% \text{ Profit } = \left(\frac{\text{selling price} - \text{cost price}}{\text{cost price}}\right)\times 100$$

$$\%\text{loss} = \left(\frac{\text{cost price} - \text{selling price}}{\text{cost price}}\right)\times 100$$

Percentage profit and loss are always calculated on cost price unless it is specifically mentioned that percentage profit or loss are to be calculated on selling price

$$\% \text{ profit } = \left[\frac{\text{selling price} - \text{cost price}}{\text{selling price}}\right]\times 100$$

$$\% \text{ loss } = \left[\frac{\text{cost} - \text{selling price}}{\text{selling price}}\right]\times 100$$

MARKED PRICE:

Marked price is also known as the list price. It is the price which is marked on the article.

$$\text{Mark up percentage} = \frac{MP - CP}{CP}\times 100$$

where CP = cost price and MP = marked price

Discount is offered on marked price.

$$\text{Discount } \% = \frac{\text{Discount}}{\text{Marked price}}\times 100$$

DISCOUNT:

A discount is offered on the marked price, therefore,

Marked Price – Discount = Selling Price.

$$\text{Discount } \% = \frac{\text{Marked Price} - \text{selling Price}}{\text{Marked Price}}\times 100$$

PARTNERSHIP:

A Partnership is an association of two or more persons who invest their money in order to carry on a certain business. A partner who manages the business is called the **working partner** and the one who simply invests the money is called the **sleeping partner**.

Partnership is of two kinds: Simple and Compound

Simple Partnership: If the capitals of the partners are invested for the same period, the partnership is called simple.

Compound Partnership: If the capitals of the partners are invested for different lengths of time, the partnership is called compound.

If the period of investment is the same for each partner, then the profit or loss is divided in the ratio of their investments.

If A and B are partners in a business, then

$$\frac{\text{Investment of A}}{\text{Investment of B}} = \frac{\text{Profit of A}}{\text{Profit of B}} = \frac{\text{Loss of A}}{\text{Loss of B}}$$

05 Average Mixture Alligation

Arithmetic Mean

Arithmetic mean of $a_1, a_2, a_3, ... a_n$

Mean = $\left[\dfrac{a_1 + a_2 + a_3 + ... + a_n}{n}\right]$.

Properties of Arithmetic Mean or average:

1. If each number is increased or decreased by a number k, the mean is also increased or decreased by the same number k.
2. If each number is multiplied or divided by a number k, the mean is also multiplied or divided by the same number k.
3. If a new data is inserted or deleted or replaced, the surplus or deficiency is uniformly distributed over the updated number of data.
4. Average of the numbers which are in Arithmetic progression is

   $\dfrac{\text{First number} + \text{Last number}}{2}$.

   For example average of the numbers 11, 13, 15, 17, ...47, 49, 51 is $\dfrac{11+51}{2} = 31$

Average Speed in a Journey:

When a person travels from A to B at a speed of x kmph and returns from B to A at the speed of y kmph, then the average speed during the journey is

$$\frac{\text{Total distance}}{\text{Total time}} = \frac{d+d}{\frac{d}{x}+\frac{d}{y}} = \frac{2xy}{x+y}$$

Note that average speed is not equal to $\dfrac{x+y}{2}$

WEIGHTED AVERAGE

Suppose there are n groups containing a_1, a_2, a_3,....a_n items and averages of the groups are x_1, x_2, x_3,....x_n, then the average of all the items in all the group put together is known as weighted average. If the weighted average is $\bar{x}$, then

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3 + ... a_nx_n}{a_1 + a_2 + a_3 + ... a_n}$$

REPEATED REPLACEMENTS

If a vessel contains x litres of wine and y litres are withdrawn and replaced with water, then y litres of the mixture is withdrawn and replaced with water, and this is continued to a total of n times, then

$$\left[\frac{\text{Final quantity of wine in the vessel}}{\text{Intitial quantity of wine in the vessel}}\right] = \left[\frac{x-y}{x}\right]^n$$

To generalize this concept, suppose initial quantity of milk = a litres, if a_1, a_2, a_3...a_n litres of the solution are withdrawn and replaced by water each time, then

$$\left[\frac{\text{Final quantity of milk in the vessel}}{\text{Intitial quantity of milk in the vessel}}\right] = \left[\frac{a-a_1}{a}\right]\left[\frac{a-a_2}{a}\right]\left[\frac{a-a_3}{a}\right].....\left[\frac{a-a_n}{a}\right]$$

06 Number Theory

NATURAL NUMBERS: Set of all positive counting numbers is known as set of natural numbers. For example {1, 2, 3, 4, 5,.....infinite}

WHOLE NUMBERS: Set of all positive counting numbers including zero is known as set of whole numbers. For example {0, 1, 2, 3, 4, 5,.....infinite}

INTEGERS: Set of all numbers which can be positive, negative or zero is known as set of integers.
For example {0, ±1, ±2, ±3, ±4, ±5,.....infinite}. Integers which are multiple of 2 are said to be even integers otherwise odd.

PRIME NUMBERS: A prime number is a natural number which has exactly two distinct natural number divisors: 1 and number itself. Prime numbers are infinite.

MERSENNE PRIME: Prime numbers of the form $2^n - 1$ are known as Mersenne primes. In order for $2^n - 1$ to be prime, it is necessary but not sufficient that n should be prime. For example, 2^3-1 is prime but $2^{11} - 1$ is not prime.

COMPOSITE NUMBERS: Any natural number which is not prime is known as composite except 1. Every composite number has more than 2 factors.

Important Points about Prime and Composite Numbers:

- ✓ The number of prime numbers are infinite
- ✓ Every prime number is odd except 2.
- ✓ Every prime number which is more than 3, can be represented as $6k + 1$ or $6k - 1$
- ✓ There are 25 prime numbers in the first 100 natural numbers.

FIBONACCI SERIES OR FIBONACCI NUMBERS:

The sequence 1, 1, 2, 3, 5, 8, 13,.... is known as Fibonacci series. If n^{th} term of the Fibonacci sequence is T_n, then $T_n = T_{n-1} + T_{n-2}$

PERFECT NUMBERS

A perfect number is defined as a positive integer which is the sum of its proper positive factors, that is, the sum of the positive divisors excluding the number itself.

The first perfect number is 6, because 1, 2, and 3 are its proper positive divisors and 1 + 2 + 3 = 6.

The next perfect number is 28. As 28 can be written as = 1 + 2 + 4 + 7 + 14.

RATIONAL NUMBER

A rational number is a number which can be expressed as a ratio of two integers. Non–integer rational numbers (commonly called fractions) are usually written as the fraction $\frac{a}{b}$, where b is not zero, a is called the numerator, and b the denominator.

Any repeating number with a constant cycle or numbers with terminating digits are also rational numbers.

CONVERSION FROM REPEATING DECIMAL TO P/Q FORM:

Note that every repeating decimal can be converted to fraction (p/q form). Suppose the number is $0.12\overline{345}$, this number is equal to 0.12345345345345......up to infinite. In this number 12 is non repeating number, 345 is repeating number, number of repeating digits is 3 and number of non-repeating digits is 2.

P/Q form is

$$\left[\frac{\text{Entire Number} - \text{Non repeating number}}{\underset{(\text{number of repeating digits})}{999999999..}\ \underset{(\text{number of non repeating digits})}{000000000}}\right]$$

For example the number $0.2\overline{34}$, 34 is repeating and 2 is non repeating. The equivalent fraction of this number is $\frac{234-2}{990} = \frac{232}{990} = \frac{116}{495}$

IRRATIONAL NUMBERS:

An irrational number is any real number that is not a rational number. It is a number which cannot be expressed as a fraction m/n, where m and n are integers, with n non–zero. The most well-known irrational numbers are π and $\sqrt{2}$.

- Sum or difference of two irrational numbers may not be irrational.
- Product or division of two irrational numbers may not be irrational.
- Product of one rational and one irrational number may or may not be irrational. For example $3\times\sqrt{2}$ is irrational but $0\times\sqrt{3}$ is rational.

LCM and HCF:

Product of two numbers = LCM × HCF

LCM is a multiple of HCF

LCM of fractions = $\left[\dfrac{\text{LCM of numerators}}{\text{HCF of denominators}}\right]$

HCF of fractions = $\left[\dfrac{\text{HCF of numerators}}{\text{LCM of denominators}}\right]$

NUMBER OF DIVISORS:

Suppose a number N can be written as $a^m \cdot b^n \cdot c^p$, where a, b, c are prime numbers and m, n, p are natural numbers. Then

Number of factors = $(m+1)(n+1)(p+1)$......

Factors are formed by taking combinations of powers of a, b and c, we know that a^0, a^1, a^2,...a^m can be the factors, similarly b^0, b^1, b^2,...b^n and c^0, c^1, c^2,...c^p. Hence total number of combinations = $(m + 1)((n + 1)(p + 1)$.

Sum of all these factors

$$= \left[\frac{a^{m+1}-1}{a-1}\right]\left[\frac{b^{n+1}-1}{b-1}\right]\left[\frac{c^{p+1}-1}{c-1}\right]....$$

NUMBER OF CO-PRIMES:

Number of positive integers less than N and co–prime to N is $N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)...$

Sum of all these numbers = $\dfrac{N.\phi(N)}{2}$

Number of ways in which a number can be represented as sum of two numbers which are co–primes to each other is $\dfrac{\phi(N)}{2}$

FERMAT'S THEOREM:

(1) If p is a prime number and a is any whole number, then the number $a^p - a$ is always multiple of p. For example, $4^{41} - 4$ is multiple of 41, $4^{71} - 4$ is multiple of 71 etc.

(2) If p is a prime number and a is any whole number, then the number $a^{p-1} - 1$ is always multiple of p where p and a are co–prime to each other.

EULER'S THEOREM:

If $\varphi(n)$ represents number of positive integers less than 'n' and co–prime to n, and a number 'a' is co-prime to n, then

$a^{\phi(n)} \equiv 1(\text{mod } n)$,

which means when $a^{\phi(n)}$ is divided by n, remainder is 1. For example, the number 100 has 40 co–primes which are less than 100 and 3 is one of them, hence we can conclude that 3^{40} when divided by 100, remainder will be 1.

WILSON'S THEOREM:

If p is a prime number, then $(p - 1)! + 1$ is always divisible by p.

For example, 18! + 1 is multiple of 19, 22! + 1 is multiple of 23 etc. For example, if we have to find the remainder when 28! is divided by 29,

We can apply this theorem; we know that 28! + 1 is multiple of 29, hence when 28! is divided by 29, remainder is –1 or 28.

Maximum power of a prime number contained in n!

Suppose we have to calculate the power of a prime number p in factorial n,

Maximum power of p in $n!$

$$= \left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+....$$

Where $\left[\dfrac{n}{p^k}\right]$ represents greatest integer function.

07 Time-Work, Time Speed Distance

Rate of work

Rate of work is defined as per day (per unit) work. If a person can finish a work in n days, then per day work of the person = $\frac{1}{n}$. Rate of work is very useful in solving questions of Time and Work.

Time taken to finish a work

The time can be calculated as $\frac{\text{Work Quantity}}{\text{Rate of work}}$

Assuming that all employees work with same efficiency, we can conclude that if the work done is constant then the number of days is inversely proportional to the number of employees working.

$M \propto \frac{1}{D}$, where M is the number of employees.

$\Rightarrow MD$ = constant

Thus we obtain a relationship

$M_1D_1 = M_2D_2$

If work is not constant, then it is directly proportional to both number of employees and number of days

$\frac{M_1D_1}{W_1} = \frac{M_2D_2}{W_2}$ = constant

Concept of Man-Days

Man days is an equivalent unit of work. For example, if we say that 20 men can do a work in 30 days, this means that total work is 20×30 man days = 600 man days, which means that the same work can be done by 10 men in 60 days or 60 men can do in 10 days only etc. So for a constant work number of man days is constant.

PIPES AND CISTERNS

This is the topic which gives the relation between the time required to fill or empty the tank with the taps opened or closed.

Till the time we have only defined the concept of positive work but in the problems related to pipes and cisterns we have to define the concept of negative work also. Concept of negative work is defined as when the work is done against the requirement.

TIME SPEED DISTANCE

Speed = $\frac{\text{Distance}}{\text{Time}}$

UNITS USED

The commonly used units of speed are Km/hr or m/s. The conversion factors of these units are given below:

Speeds in m/s = $\frac{5}{18} \times \text{speed in km/hr}$

or speed in km/hr = $\frac{18}{5} \times \text{Speed in m/s}$

For example a speed of 72 km/hr is equivalent to $\frac{5}{18} \times 72$ = 20 m/s or speed of 40 m/s = $\frac{18}{5} \times 40$ =144 km/hr.

INVERSE AND DIRECT RELATIONS OF SPEED, TIME AND DISTANCE:

(a) *When distance is constant, time is inversely proportional to speed.*

(b) *When the time is constant, then the distance covered is directly proportional to the time taken.*

CROSSING OF TWO TRAINS

If two trains are running in the same directions, their speeds are v_1, v_2 and their length are ℓ_1, ℓ_2.

Time taken by faster train to overtake the slower train

$$T = \frac{\ell_1 + \ell_2}{v_1 - v_2}$$

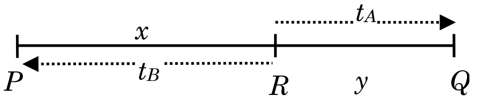
In opposite direction time taken to cross each other

$$T = \frac{\ell_1 + \ell_2}{v_1 + v_2}$$

Time taken by a train of length ℓ_1, to cross the platform of length ℓ

$T = \frac{\ell_1 + \ell}{v}$, where v is the speed of the train.

■ **SPECIAL CASE:** Suppose the two persons A and B start moving simultaneously from the points P and Q respectively. They meet at a point R after a time T and take t_A and t_B time to reach the point Q and P respectively.



Let the distances PR and RQ are x and y and speeds of A and B are V_A and V_B, then

$x = V_A \times T$ (1)

$y = V_B \times T$ (2)

$x = t_B \times V_B$ (3)

$y = t_A \times V_A$ (4)

Now from the above equations,

$$\therefore \frac{V_A}{V_B} = \sqrt{\frac{t_B}{t_A}} \text{ and } T = \sqrt{t_A t_B}$$

Average speed: Average speed is always defined as total distance covered in unit time. ***It is not always equal to average of the speeds.***

$$\text{Average speed} = \left[\frac{\text{total distance}}{\text{total time}}\right]$$

For example if a man goes to office @ 20 km/hr and comes back @ 30 km/hr then the average speed will not be equal to $\frac{20+30}{2} = 25$

$$\text{Average speed} = \frac{x+x}{\frac{x}{20}+\frac{x}{30}}$$

$$= \frac{2\times 20 \times 30}{20+30} = 24\,\text{km/h}$$

BOATS AND STREAMS:

When a boat is moving in the same direction as the stream or water current, it is said to be moving *with the stream or current.*

When a boat is moving in a direction opposite to that of the stream or water current, it is said to be moving *against the stream or current.*

If the boat is moving with a certain speed in water that is not moving, the speed of the boat is then called *speed of the boat in still water.*

Speed of the boat against stream = Speed of the boat in still water – Speed of the stream

Speed of the boat with stream = Speed of the boat in still water + Speed of the stream

These two speeds, the speed of the boat against the stream and the speed of the boat with the stream, are RELATIVE SPEEDS.

If x is the speed of the boat in still water and y is the speed of the stream, then

Speed of the boat with the stream = $(x+y)$

Speed of the boat against stream = $(x-y)$

RACES AND CIRCULAR TRACKS

Suppose there are two individuals, A and B, participating in a race. If A begins the race and B starts 10 seconds later, we say that A has a "head start" of 10 seconds. Similarly, if A starts the race first and covers a distance of 10 meters before B starts, we say that A has a "head start" of 10 meters in the race against B.

In a race between A and B where B is the winner, by the time B reaches the winning post, if A still has another 10 meters to reach the winning post, then we say that B has won the race by 10 meters. Similarly, if A reaches the winning post 10 seconds after B reaches it, then we say that B has won the race by 10 seconds.

■ When two persons are running around a circular track in OPPOSITIE directions the relative speed is equal to the sum of the speeds of the two individuals and from one meeting point to the next meeting point, the two of them TOGETHER cover a distance equal to the length of the track.

■ When two persons are running around a circular tack in the SAME direction the relative speed is equal to the difference of the speeds of the two individuals and from one meeting point to the next meeting point, the faster person covers one COMPLETE ROUND more than the slower person.

We can now tabulate the time taken by the persons to meet for the first time ever or for the first time at the starting point in various cases.

■ WHEN TWO PEOPLE ARE RUNNING AROUND A CIRCULAR TRACK

Let the two people A and B with respective speeds of u and $v\,(u > v)$ be running around a circular track of length d, starting at the same point and at the same time. Then, the following table gives the details about the time taken in their meeting under various conditions.

	Running in the **same** directions	Running in the **opposite** directions
Time taken to meet for the first time ever	$\frac{d}{u-v}$	$\frac{d}{u+v}$
Time taken to meet for the first time at the starting point	LCM of $\left\{\frac{d}{u},\frac{d}{v}\right\}$	LCM of $\left\{\frac{d}{u},\frac{d}{v}\right\}$

To determine the time, it takes for two persons to meet at the starting point for the first time, we need to calculate the time each person takes to complete one full round and then find the least common multiple (LCM) of these two times. The times taken by the two individuals to complete one full round are $\frac{d}{u}$ and $\frac{d}{v}$ respectively.

08 Geometry

Angles associated with parallel lines

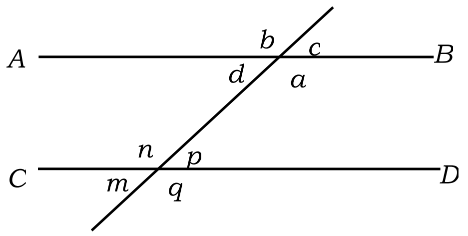
Vertically opposite angles, Corresponding angles, alternate angles are explained by this diagram.

In this diagram

$b = a$ (Vertically opposite angles)

$b = n$ (Corresponding angles)

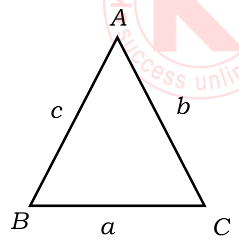
$d = p$ (Alternate angles)



Triangular Inequalities:

Sum of any two sides is more than the third and difference of any two sides is less than the third.

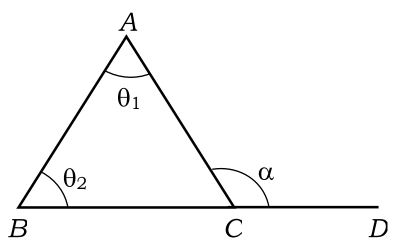
$a + b > c$, $b + c > a$ and $c + a > b$



Exterior Angle

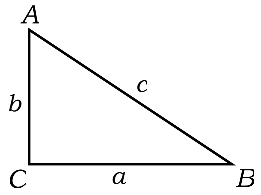
Again extending any of the side, we get external angle. Exterior angle is equal to sum of 2 opposite interior angles.

$\alpha = \theta_1 + \theta_2$



PYTHAGOREAN THEOREM

In a right angled triangle square of hypotenuse is equal to sum of squares of base and height.



Let ABC be a right angled triangle with side lengths a, b, and c, then

$a^2 + b^2 = c^2$.

Pythagorean Triplets

A set of three natural numbers a, b and c which satisfy the condition $a^2 + b^2 = c^2$ are known as Pythagorean triplets. For example:

(3, 4, 5), (5, 12, 13), (7, 24, 25), (8, 15, 17) etc.

Acute Angled Triangle

If all the angles in a triangle are less than 90°, then the triangle is known as acute angled triangle. A triangle with sides a, b, c is an obtuse angled triangle if

$a^2 < b^2 + c^2$, $b^2 < a^2 + c^2$ and $c^2 < a^2 + b^2$

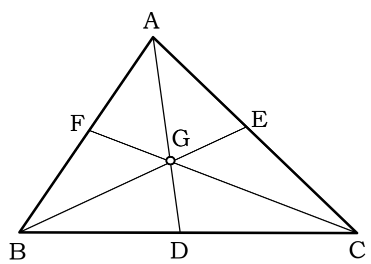
Obtuse Angled Triangle

If one angle in a triangle is obtuse, then triangle is known as obtuse angled triangle. A triangle with sides a, b, c is an obtuse angled triangle if

$a^2 > b^2 + c^2$, $b^2 > a^2 + c^2$ or $c^2 > a^2 + b^2$

MEDIAN

In the triangle shown below AD, BE and CF are medians. These medians intersect at a common point G, known as centroid of the triangle.



Also medians intersect each other in the ratio 2 : 1.

$$\frac{AG}{GD}=\frac{BG}{GE}=\frac{CG}{GF}=\frac{2}{1}$$

Length of a Median

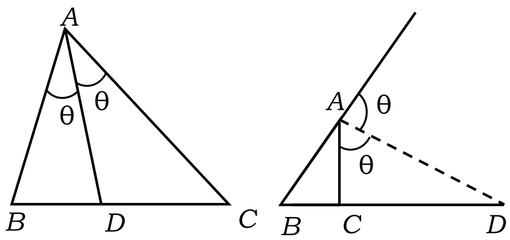
In the triangle ABC (shown above), D is the midpoint and AD is the median.

By Apollonius theorem length of AD is given by

$AB^2 + AC^2 = 2\ (AD^2 + DC^2)$

ANGLE BISECTOR THEOREM

In any triangle ABC, the angle bisector of A divides the line BC in the ratio of AB and AC. This theorem is valid for internal as well as external angle bisector



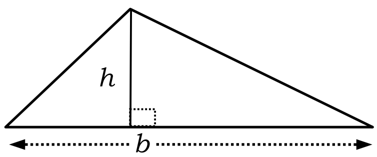
$$\frac{AB}{AC}=\frac{BD}{DC}$$

Length of the angle bisector:

The length of the internal angle bisector AD is given by $AD^2 = AB \times AC - BD \times DC$

AREA OF A TRIANGLE

1. Area = $\frac{1}{2} \times b \times h$



2. If height is not given but all the sides are given, then area of the triangle can be calculated by Heron's Formula

   Area = $\sqrt{s(s-a)(s-b)(s-c)}$

   where s is semi perimeter and $s = \frac{a+b+c}{2}$

   and a, b, c are the sides of triangle.

3. When two sides and angle between them is given, if the two sides are a,b and angle between them is θ, then

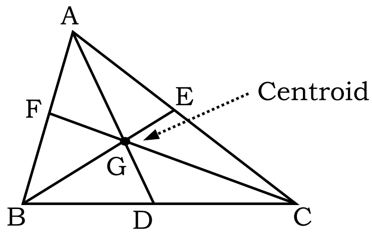
   Area = $\frac{1}{2} ab \sin\theta$

4. If all side of triangle are equal (Equilateral triangle),

   Area = $\frac{\sqrt{3}}{4}a^2$

CENTROID

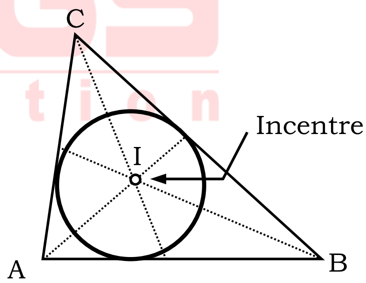
Centroid is the intersection point of three medians of a triangle. All the three median are concurrent and they divide each other in the ratio 2:1.



$$\frac{AG}{GD}=\frac{BG}{GE}=\frac{CG}{GF}=\frac{2}{1}$$

INCENTRE

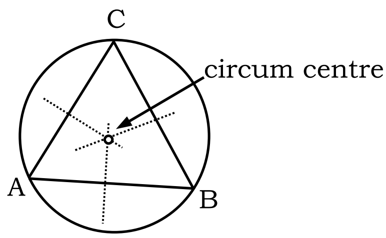
Incentre is the intersection point of the angle bisectors of three internal angles of a triangle. Incentre always lies inside the triangle.



A circle that touches all the three circles internally is known as incircle.

CIRCUMCENTRE

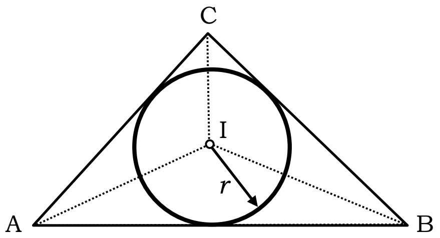
Circumcentre is the intersection point of the three perpendicular bisectors of a triangle. This centre is equidistance from the three vertices.



INCIRCLE

In circle is the circle that is inscribed in a given triangle. Centre of the in circle is the intersection point of three angle bisectors of internal angles of a triangle. If the sides of the triangle are *a, b* and c and *r* is in radius of the triangle

$r = \frac{\Delta}{s}$

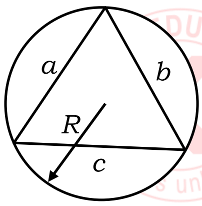


where Δ is area of the triangle and $s = \frac{a+b+c}{2}$

If the triangle is equilateral of side *a*, then the value of in radius = $\frac{a}{2\sqrt{3}}$

CIRCUMCIRCLE:

The circumcircle of a triangle is the unique circle passing through the three vertices of the triangle. Its center is called the circumcenter.



Circum-radius, $R = \frac{abc}{4\Delta}$

Where *R* is the circum-radius and *a, b, c* are three sides.

If the triangle is equilateral of side *a*, then the value of circum-radius = $\frac{a}{\sqrt{3}}$

Similar Figures:

Two figures are said to be similar, if they look alike or their shape is same. Their size may be different.

Similar Triangles:

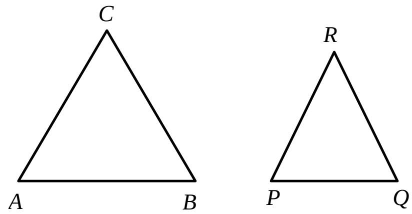
Two triangles are said to be similar if they are in the same shape, now same shape can be decided by different measures

1. The three angles of one triangle are equal to the three angles of the second triangle. The corresponding angles of two similar triangles are equal but the corresponding sides are only proportional and not equal.
2. Two sides of one triangle are proportional to two sides of the other and the included angles are equal,
3. The three sides of one triangle are proportional to the three sides of another triangle.

In two similar triangles,

(1) Side opposite to similar angles bears a constant ratio. Ratio of corresponding sides = ratio of perimeters.

(2) Since area is proportional to square of the side, hence Ratio of areas = Ratio of squares of corresponding sides for example if ratio of the corresponding sides of two triangles is 1:2, ratio of the corresponding area will be 1:4



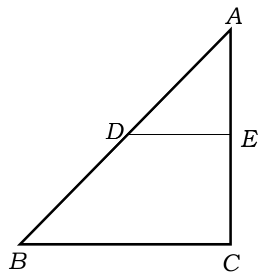
In the above diagram there are two triangles, suppose $\angle A = \angle P$ and $\angle B = \angle Q$, then triangle become similar. Now,

$\frac{BC}{AC} = \frac{QR}{PR}$ and $\frac{BC}{AB} = \frac{QR}{PQ}$

Ratio of the sides opposite to the corresponding same angles will be same.

Mid-Point Theorem

In any triangle if we join midpoint of any two sides, then this line will be parallel to the third side and half of the length of the third side.



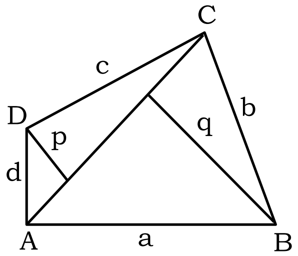
In this triangle *ADE* and *ABC* are similar triangles. Hence *DE* is parallel to *AB.*

As D and E are the mid points,

$\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC} = \frac{1}{2}$

Quadrilaterals

Any four sided closed figure is called a Quadrilateral. The sum of four angles of a Quadrilateral is equal to 360°. The perpendicular drawn to a diagonal (in a Quadrilateral) from the opposite vertices are called offsets.



Suppose the length of AC is ℓ, then

Area of the quadrilateral is $\frac{\ell}{2}(p+q)$

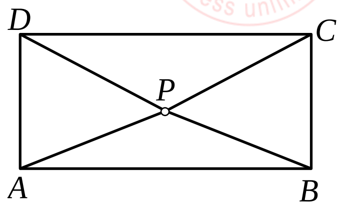
SQUARE

A square is a special rectangle whose length and breadth are same. Diagonals of a square are perpendicular bisectors.

Area of a square = $(\text{side})^2$ and diagonal = $\sqrt{2} \times side$

RECTANGLE

A rectangle is a parallelogram in which two adjacent angles are equal or each angles is equal to 90°.

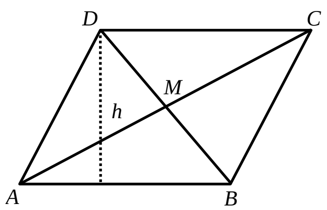


1. If P is any point inside the rectangle, then

   $PA^2 + PC^2 = PB^2 + PD^2$
2. Area of a rectangle = ab
3. If 'a' and 'b' are the two adjacent sides the diagonal is given by $\sqrt{a^2+b^2}$

PARALLELOGRAM

A Quadrilateral in which opposite side are parallel is called a parallelogram.

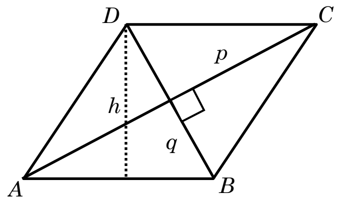


Area of the parallelogram = Base × height = $AB \times h$

Area = $AB \times AD \times \sin\theta$, where θ is the angle between AB and AD.

RHOMBUS:

A rhombus is a parallelogram in which all four sides are equal and all four angles may not be equal.



If diagonals are p and q, and side is a then $p^2 + q^2 = 4a^2$

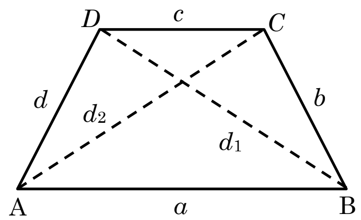
Area of the rhombus = $\frac{1}{2}pq$, where p and q are the diagonals.

Area can also be calculated as base × height = $AB \times h$

TRAPEZIUM

If one pair of opposite sides of a quadrilateral is parallel, then it is called a trapezium.

The two sides other than the parallel sides in a trapezium are celled as oblique sides.



Area of trapezium = $\frac{1}{2}(a+c)\times h$, where h is the distance between a and c.

Diagonals are d_1 and d_2,

$d_1^2 + d_2^2 = b^2 + d^2 + 2ac$

POLYGONS:

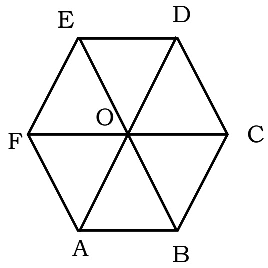
The sum of interior angles of a convex or concave polygon is = $(n-2)\times 180$, where n is the number of the sides of the polygon.

The sum of the external angles of a polygon is equal to 360°

For a regular polygon value of each interior angle is $\frac{(n-2)\times 180}{n}$ and value of exterior angle is $\frac{360}{n}$.

Regular Hexagon

Regular hexagon consists of six identical line segments. Each internal angle is same and equal to 120°



Diagonal $AC = \sqrt{3}a$

Diagonal $AD = 2a$.

Areas = $6 \times \left(\frac{\sqrt{3}}{4}a^2\right)$

Radius of the circumcircle = a

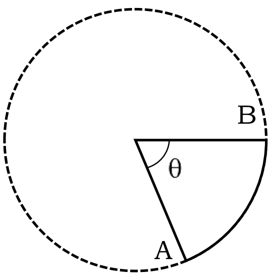
Number of diagonals in a polygon $= \frac{n(n-3)}{2}$

Area of any regular polygon whose side is a,

$= \frac{na^2}{4}\cot\left(\frac{180}{n}\right)$

ARC

In the given diagram AB is the arc.

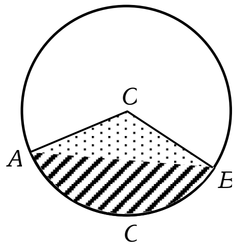


If the angle subtended by the $arc = \theta$, then the arc length = $r\theta$

or $\theta = \frac{\text{Arc Length}}{\text{Radius}}$

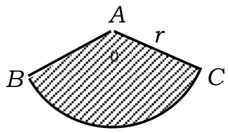
Area of sector and segment:

A sector is a region bounded by two radii and an arc lying between the radii and a **segment** is a region bounded by a chord and an arc lying between the chord's endpoints.



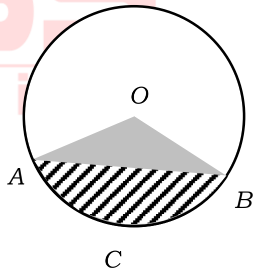
In this diagram the complete region *OACBO* is sector and the region *ACBA* is segment.

If the angle of the sector is θ, then



Area of the sector $= \frac{\theta}{360}\pi r^2$

Area of the segment = area of the sector – area of triangle



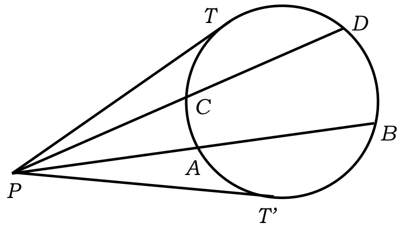
If the angle $AOB = \theta$ and radius of the circle = r, then

Area of the segment $= \frac{\theta}{360}\pi r^2 -$ area of triangle *OAB*

Area of segment = $\left[\frac{\theta}{360}\pi r^2 - \frac{1}{2}r^2\sin\theta\right]$

Secant Lines:

If *PAB* and *PCD* are two secants lines, then



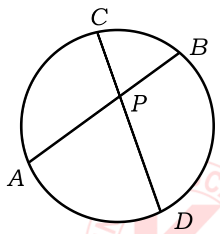
If *PAB* and *PCD* are two secants, then

$PA \times PB = PC \times PD$

A line that touches the circle at only one point is a tangent to the circle *PT* is a tangent

$PA.PB = PC.PD = PT^2$.

Two tangents can be drawn to the circle from any point outside the circle and these two tangents are equal in length *P* is the external point and the two tangents *PT* and *PT'* are equal.

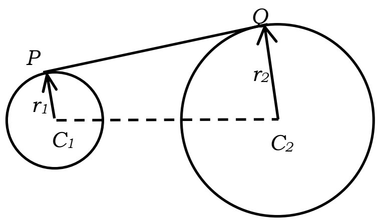


If the point *P* is inside the circle, then again the same rule can be applied,

In the above diagram ***PA.PB = PC.PD***

Direct Common Tangent

A direct common tangent is a tangent that touches both the circle on the same side. In this diagram radii of the circles are r_1 and r_2 and distance between the centres is *d*,

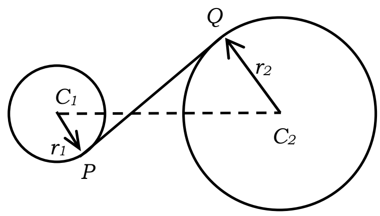


Length of the direct common tangent

$PQ = \sqrt{d^2 - (r_2 - r_1)^2}$

Transverse Common Tangent

A transverse common tangent is a line that touches both the circles where the points of contacts lie on the opposite side of the line joining the centres of both the circles.

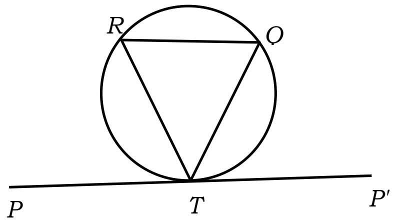


In the given diagram *PQ* is the transverse common tangent.

$PQ = \sqrt{d^2 - (r_2 + r_1)^2}$

Alternate segment theorem:

If *PTP'* is a tangent, *Q* and *R* are any two points on the circle. Then,

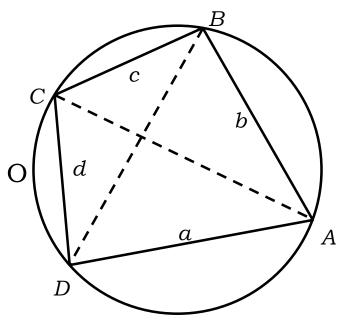


$\angle QRT = \angle QTP'$

$\angle RQT = \angle RTP$

Cyclic Quadrilateral

Any quadrilateral whose all four vertices lie on a circle, is known as cyclic quadrilateral

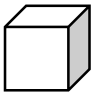
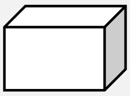
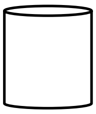
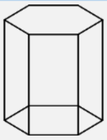
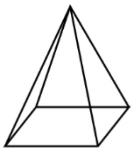
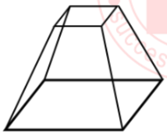
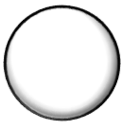


1. Sum of opposite angles = 180, that means $\angle A + \angle C = \angle B + \angle D = 180$
2. Area of quadrilateral *ABCD*

   $= \sqrt{(s-a)(s-b(s-c)(s-d)}$, where

   $s = \dfrac{a+b+c+d}{2}$
3. If the lengths of the diagonals are *p* and *q*, then
   pq = ac + bd

Volume and Surface Areas

Solid		Volume	Lateral Surface area	Total Area
Cube		a^3, where a is the side of the cube	---	$6a^2$
Cuboid		abc, where a, b and c are the three sides	---	$2(ab+bc+ca)$
Cylinder		$\pi r^2 h$, where r is the radius and h is the height.	$2\pi rh$	$2\pi rh+2\pi r^2$
Prism		$A \times h$, where A is the base area and h is the height.	$P \times h$, where P is the perimeter of the base.	Lateral Area + Area of the Top and Bottom
Pyramid		$\frac{A \times h}{3}$, where A is the area of the base and h is the perpendicular height.	$\frac{1}{2} \times P \times \ell$, where P is the perimeter of the base and ℓ is the slant height	Lateral Area + Area of the Bottom
Cone		$\frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height	$\pi r \ell$, where ℓ is the slant height and $\ell^2 = h^2 + r^2$	$\pi r \ell + \pi r^2$
Frustum of Pyramid		$\frac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right) \times h$ Where A_1, A_2 are the areas of the bottom and top, h is the height	$\frac{1}{2}(P_1 + P_2) \times \ell$, where P_1, P_2 are the perimeters of the base and top, ℓ is the slant height.	Lateral Area + Area of the top and bottom
Frustum of cone		$\frac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right) \times h$ Where A_1, A_2 are the areas of the bottom and top, h is the height	$(\pi r_1 + \pi r_2) \times \ell$, where r_1, r_2 are the radii of the base and top, ℓ is the slant height.	Lateral Area + Area of the top and bottom
Sphere		$\frac{4\pi}{3} r^3$, where r is the radius of the sphere.	---	$4\pi r^2$
Hemisphere		$\frac{2\pi}{3} r^3$, where r is the radius of the sphere.	$2\pi r^2$	$2\pi r^2 + \pi r^2$
Ring		$\frac{\pi(r_1 + r_2)(r_2 - r_1)^2}{4}$, where r_1, r_2 are the radii of the inner and outer circles.	---	$\pi^2(r_2^2 - r_1^2)$

09 Simple and Compound Interests

Simple interest (SI) = Principal × Time × Rate

$$SI = \frac{P \times R \times T}{100}$$

Where P is principal, T is the time period and R is the interest rate.

Amount = Principal + Interest

If the interest rates are R_1, R_2 and R_3 for the time periods T_1, T_2 and T_3. if the principal money is P, then the total amount after $T_1 + T_2 + T_3$ time,

$$= P + P\left[\frac{R_1 \times T_1}{100} + \frac{R_2 \times T_2}{100} + \frac{R_3 \times T_3}{100}\right]$$

Compound Interest

If P is principal, T is the time period, R is the interest rate, then the total amount after T years

$$A = P\left(1 + \frac{R}{100}\right)^T$$

And the compound interest (CI)

$$CI = A - P = P\left(1 + \frac{R}{100}\right)^T - P$$

Compound interest reckoned half–yearly

If the annual rate is $r\%$ per annum and is to be calculated for n years, then in this case, rate is $\frac{r}{2}\%$ half yearly and time $(2n)$ half years.

Final amount after $2n$ half years,

$$A = P\left[1 + \frac{r/2}{100}\right]^{2n}$$

Compound interest reckoned quarterly

In this case, rate = $\frac{r}{4}\%$ quarterly and time = $(4n)$ quarter years.

$$A = P\left[1 + \frac{(r/4)}{100}\right]^{4n}$$

If interest rate is different for different intervals, then the value of the amount after n such intervals

will be:

$$P\left(1 + \frac{r_1}{100}\right)\left(1 + \frac{r_2}{100}\right)....\left(1 + \frac{r_n}{100}\right)$$

Suppose the interest rate per annum is $r\%$ and interest is reckoned per day (or per hour), then number of intervals become very large i.e. = 365, and interest per interval can be takes as $r/365$. Thus the net amount after one year will be

$$P\left(1 + \frac{r/365}{100}\right)^{365} = Pe^{\frac{r}{100}}$$

If the time is not integer and it is of the form of $n + f$, then the value of a principal amount P, at the rate of r% compound interest will be:

$$P\left(1 + \frac{r}{100}\right)^n\left(1 + \frac{fr}{100}\right)$$

Compound interest generated in n^{th} interval can be written as:

$$CI_{n^{th}\text{ year}} = P\left(1 + \frac{r}{100}\right)^{n-1} \times \frac{r}{100}$$

Depreciation of asset

The value of a machine or any other article subject to wear and tear, decreases with time. This decrease is called its depreciation. Thus if P_0 is the value at a certain time and $r\%$ per annum is the rate of depreciation per year, then the value P_1 at the end of t years is:

$$P_1 = P_0\left(1 - \frac{r}{100}\right)^t$$

10 Quadratic Equations

Solution of quadratic Equation:

The roots of the quadratic equation $ax^2+bx+c=0$ are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Nature of roots:

The roots of the quadratic equation $ax^2+bx+c=0$ are:

Real and distinct if $b^2-4ac>0$

Real and equal if $b^2-4ac=0$

Complex numbers if $b^2-4ac<0$

Roots are real and rational if b^2-4ac is a perfect square and a, b, c are also rational numbers.

Sum and Product of the roots:

If the roots of the quadratic equation $ax^2+bx+c=0$ are α and β, then

$$\text{Sum}=\alpha+\beta=-\frac{b}{a}$$

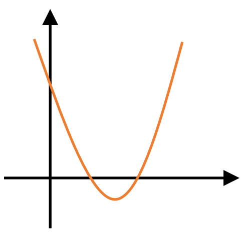
$$\text{Product}=\alpha\beta=\frac{c}{a}$$

GRAPHS OF QUADRATIC FUNCTIONS:

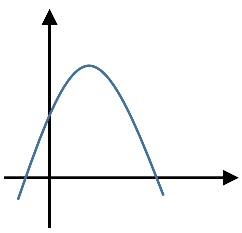
The graphs of all quadratic functions produce U shaped graphs called **parabolas**, which open either upwards or downwards.

If $a>0$, the graph opens upwards

If $a<0$, the graph opens downwards



$y=ax^2+bx+c$
$(a>0)$



$y=ax^2+bx+c$
$(a<0)$

Maximum and Minimum value of ax^2+bx+c

The expression is minimum when $a>0$

The minimum value is $\frac{4ac-b^2}{4a}$

The expression is maximum when $a<0$

The maximum value is $\frac{4ac-b^2}{4a}$

Both minimum and maximum values occur at $x=-\frac{b}{2a}$

Formation of equation:

If the sum and product of the roots of a quadratic equation are S and P, then the equation can be written as

$$x^2-Sx+P=0$$

for example, equation whose roots are 3 and 4 can be written as $x^2-(3+4)x+3\times4=0$, that is $x^2-7x+12=0$

Cubic equations

An equation of the form $ax^3+bx^2+cx+d=0$ is a cubic equation. If the roots of this equation are α, β and γ, then

$$\alpha+\beta+\gamma=-\frac{b}{a}$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$$

$$\alpha\beta\gamma=-\frac{d}{a}$$

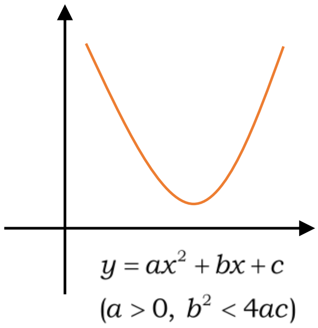
Some special results

- If one of the root of a quadratic equation is $u+\sqrt{v}$, then the other is $u-\sqrt{v}$, provided that a, b, c are rational numbers.
- Complex roots always occur in pairs, provided that a, b, c are real numbers.
- If $\sqrt{n+\sqrt{n+\sqrt{n+\ldots\infty}}}=x$, then $x(x-1)=n$
- If $\sqrt{n-\sqrt{n-\sqrt{n-\ldots\infty}}}=y$, then $y(y+1)=n$
- If $\sqrt{n\sqrt{n\sqrt{n\sqrt{n\ldots\infty}}}}=z$, then $z=n$

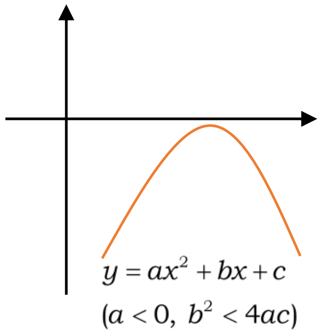
Sign of the expression ax^2+bx+c

The sign of the expression ax^2+bx+c depends upon the shape of the graph $y=ax^2+bx+c$.

- The sign is always positive, when the graph is vertically upwards and it does not intersect with x axis.



- The sign is always Negative, when the graph is vertically downwards and it does not intersect with x axis.



11 Indices – Surds – Logarithm

LAWS ON INDICES:

If a is a non-zero real number and m, n are real numbers, then

1. $(a^m)(a^n) = a^{m+n}$

2. $\frac{a^m}{a^n} = a^{m-n}$, in this result if we put m = n, then we get $\frac{a^m}{a^m} = a^{m-m}$ or $a^0 = 1$. Thus any non-zero real number to the power 0 is 1.

3. $(a^m)^n = a^{mn}$, this can be proved by writing it in the following manner,

$a^m.a^m.a^m........n\ times = a^{m+m+m+...n\ times} = a^{mn}$

For example $(2^5)^4 = 2^{20}$, but $2^{5^4} \neq 2^{20}$.

4. $a^{-m} = \frac{1}{a^m}$, for example $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

5. $a^{m/n}$ is n^{th} root of a^m, m is any integer and n is any positive integer.

6. For the solution of $a^x = a^y$ where a, x, y are real numbers, consider the following cases carefully.

(1) If $a = 1$, then x, y may be any real numbers

(2) If $a = -1$, then x, y may be both even or both odd.

(3) If $a = 0$, then x, y may be any non–zero real numbers of same sign

(4) If $a \neq \pm 1$ or 0, then $x = y$

7. For the solution of $a^x = b^x$, a, b, x are real numbers. Consider the following cases carefully:

(1) If $a \neq \pm b$, then $x = 0$

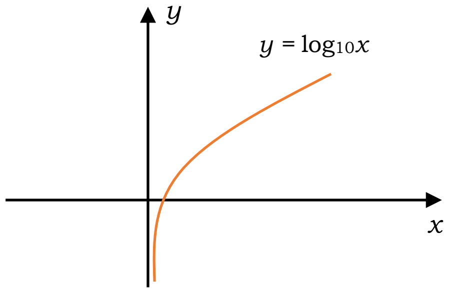
(2) If $a = b \neq 0$, then x may have any real value

(3) If $a = -b$, then x is even.

8. Approximate value of $\left(1+\frac{1}{n}\right)^n$ is **2.7**, where n is a large number.

LOGARITHMS

If $\log_a m = x$, then $a^x = m$, where $m > 0$, $a > 0$ and $a \neq 1$



Graph of $\log_{10} x$

LAWS OF LOGARITHMS

If $a > 0$ and $a \neq 1$, then

1. $\log_a(m.n) = \log_a m + \log_a n$, where m and n are positive numbers.

2. $\log_a m^n = n \log_a m$, where m is positive.

3. $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$, where m and n are positive numbers.

4. For any positive real numbers r and b, $b \neq 1, \log_b r = \frac{\log_a r}{\log_a b}$

5. $\log_a b = \frac{1}{\log_b a}$, where $b > 0, b \neq 1$ and $a > 0$ but not equal to 1

6. $\log_{a^k} m^n = \frac{n}{k} \log_a m$

7. $a^{\log_a x} = x$, where x is a positive number.

8. $a^{\log_c x} = x^{\log_c a}$ for any x and c, which are positive and not equal to 1. For example $3^{\log_4 5} = 5^{\log_4 3}$

9. $\log_a x = \log_a y \Rightarrow x = y$

10. $\log_a x = \log_b x \Rightarrow a = b$ or $x = 1$

11. $\log_a x > \log_a y \Rightarrow \begin{cases} x > y & \text{if } a > 1 \\ x < y & \text{if } 0 < a < 1 \end{cases}$

SURDS:

A surd is defined as irrational root of a rational number, for example $\sqrt{2}, \sqrt{3}$ etc are surds

SQUARE ROOT OF A SURD:

Suppose the surd is $a+\sqrt{b}$ and it's square root is $\sqrt{x}+\sqrt{y}$, then $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$,

Squaring both the sides,

$a+\sqrt{b}=x+y+2\sqrt{xy}$

So $x+y=a,\ 2\sqrt{xy}=\sqrt{b}$

Take the following example,

$\sqrt{8+2\sqrt{15}}=\sqrt{8+2\sqrt{5}\sqrt{3}}=\sqrt{5+3+2\sqrt{5}\sqrt{3}}$

$\Rightarrow x=5, y=3$ and the square root is $\sqrt{5}+\sqrt{3}$

Some important results:

- If $\sqrt{a+\sqrt{a+\sqrt{a+.......\infty}}}=n$, then $n(n-1)=a$.

  For example $\sqrt{12+\sqrt{12+\sqrt{12+....\infty}}}=n$, then $n(n-1)=12$ or $n=4$.

- If $\sqrt{a-\sqrt{a-\sqrt{a-.......\infty}}}=n$, then $n(n+1)=a$.

  For example $\sqrt{12+\sqrt{12+\sqrt{12+....\infty}}}=n$, then $n(n+1)=12$ or $n=3$.

- If $\sqrt{a+\sqrt{a-\sqrt{a+\sqrt{a-\sqrt{a+....\infty}}}}}=n$, then $n(n-1)=(a-1)$. For example $\sqrt{21+\sqrt{21-\sqrt{21+\sqrt{21-....\infty}}}}=n$, then $n(n-1)=20$ or $n=5$.

- If $\sqrt{a-\sqrt{a+\sqrt{a-\sqrt{a+\sqrt{a-....\infty}}}}}=n$, then $n(n+1)=(a-1)$. For example $\sqrt{21+\sqrt{21-\sqrt{21+\sqrt{21-....\infty}}}}=n$, then $n(n+1)=20$ or $n=4$.

12 Series and Sequences

SOME BASIC SERIES:

1. The series 1 + 2 + 3 +...n is written as

$\sum n$ and it is equal to $\frac{n(n+1)}{2}$

2. $\sum\left(\sum n\right) = \sum 1 + \sum 2 + \sum 3 + ... \sum n$

= (1) + (1 + 2) + (1 + 2 + 3) + ... n terms

$= \left(\frac{n(n+1)(n+2)}{3 \times 2 \times 1}\right)$

3. Sum of the first n odd numbers is n^2 and sum of the first n even numbers is $n(n + 1)$.

1 + 3 + 5 + ... n terms = n^2

2 + 4 + 6 + ... n terms = $n(n + 1)$

4. Sum of the squares of the first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$ and sum of the cubes of the first n natural numbers is $\left\{\frac{n(n+1)}{2}\right\}^2$

$\sum n^2 = 1^2 + 2^2 + ...n^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum n^3 = 1^3 + 2^3 + ...n \text{ terms} = \left(\frac{n(n+1)}{2}\right)^2$

5. $1.2 + 2.3 + 3.4 + ...n$ terms $= \frac{n(n+1)(n+2)}{3}$

$1.2.3 + 2.3.4 + 3.4.5....n$ terms $= \frac{n(n+1)(n+2)(n+3)}{4}$

Arithmetic Progression:

If the first term and common difference of an Arithmetic Progression are a and d, then n^{th} term of the series is given by

$T_n = a + (n-1)d$

Sum of the first n terms is given by

$S = \frac{n}{2}[2a + (n-1)d]$

This formula can also be written as

$S = \frac{n}{2}[a + \ell] = \frac{n}{2}[\text{first term} + \text{last term}]$

where ℓ is the last term of the series.

Arithmetic Mean:

Arithmetic Mean of any series is defined as

$\left(\frac{\text{sum}}{\text{Number of terms}}\right)$

For any Arithmetic Progression,

Arithmetic mean (AM) $= \frac{a+\ell}{2}$

If the arithmetic mean of an Arithmetic Progression is AM and there are n terms in the series, then

Sum = AM $\times n$

If $a_1, a_2, a_3, ...a_n$ are in Arithmetic Progression, then

$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = ... = 2\text{AM}$

Assumptions of terms in AP:

While assuming the terms we should keep one thing in our mind that assumption of the terms should be such which makes the calculations easy.

Three terms $a-d, a, a+d$

Four terms $a-3d, a-d, a+d, a+3d$

Five terms $a-2d, a-d, a, a+d, a+2d$

Geometric Progression

In a GP, he first term is non zero and each successive term is r times the preceding term, i.e. $\frac{t_n}{t_{n-1}} = r$ is a constant. In general, if the first term of a *G.P.* is a and the common ration is r, then n^{th} term is

$T_n = ar^{n-1}$

The sum of the first n terms of a *G.P.* is given by

$S_n = a\left(\frac{r^n - 1}{r-1}\right)$ if $r > 1$

$S_n = a\left(\frac{1-r^n}{1-r}\right)$ if $r < 1$

If the *GP* is an infinite *GP* with common ratio r (where $-1 < r < 1$), then sum is given by:

$S = \frac{a}{1-r}$

Assumptions of terms in *GP:*

While assuming the terms we should keep one thing in our mind that assumption of the terms should be such which makes the calculations easy.

Three terms $\frac{a}{r}, a, ar$

Four terms $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

Five terms $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

Geometric Mean:

The Geometric mean G of any two numbers a and b is given by $\sqrt{ab}$ where the terms a, G, b are in *G.P.*

Geometric Mean of a and $b = \sqrt{ab}$

If a_1, a_2, a_3, ..., a_n are n numbers, then the Geometric Mean G, of these numbers is given by:

$G = (a_1 a_2 a_3 .... a_n)^{1/n}$

If $a_1, a_2, a_3, ...a_n$ are in Geometric Progression, then

$a_1 \cdot a_n = a_2 \cdot a_{n-1} = a_3 \cdot a_{n-2} = ... = \text{GM}^2$

Harmonic Mean:

The harmonic mean H of any two numbers a and b is a number such that the numbers $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in *A.P.* Thus, we have $\frac{2}{H} = \frac{1}{a} + \frac{1}{b}$ or

Harmonic Mean, $H = \frac{2ab}{a+b}$

Relation in AM, GM and HM:

For any given series of positive numbers,

AM ≥ GM ≥ HM.

For any two numbers, a and b and their *AM, GM* and *HM* are *A, G* and *H*, then

$G^2 = A \times H$

13 Permutations and Combinations

Product Rule:

The product rule states that if there are m ways to do one thing, and n ways to do another, then there are $m \times n$ ways to do both. This can be extended to any number of independent events.

The general formula for the product rule in permutations is:

$m_1 \times m_2 \times m_3 \times .... \times m_n$

Where $m_1, m_2, ..., m_n$ are the number of ways that each independent event can occur.

Important formulae:

✓ $^nC_r = \frac{n!}{(n-r)!\,(r!)} = \frac{n(n-1)(n-2)....(n-r+1)}{r!}$

✓ $^nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)....(n-r+1)$

✓ $^nC_r + {}^{n+1}C_r = {}^{n+1}C_{r+1}$

✓ $^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$

✓ Permutation of n distinct objects without repetition = $n!$

✓ Permutation of n distinct objects when repetition is allowed = n^n

✓ Permutation of mixed objects
$= \frac{n!}{(a_1!)(a_2!)(a_3!).....(a_k!)}$
where $a_1, a_2, ..., a_k$ are the frequencies of the identical objects.

Example: How many distinct permutations can be formed using the letters of the word 'BOOKKEEPER'?

Solution: There are 10 letters and there are 2 O's, 2 K's and 3 E's.

Number of permutations

$$= \frac{\overset{\text{Total 10 letters}}{10!}}{\underset{\text{B's}}{1!} \times \underset{\text{O's}}{2!} \times \underset{\text{K's}}{2!} \times \underset{\text{E's}}{3!} \times \underset{\text{P's}}{1!} \times \underset{\text{R's}}{1!}} = \frac{10!}{2! \times 2! \times 3!}$$

Circular Permutations:

Circular permutation of n distinct objects $= (n-1)!$

Circular Permutation of r objects taken from n distinct objects $= {}^nC_r \times (r-1)! = \frac{n!}{(n-r)! \times r}$

Number of triangles that can be formed using n straight lines = $^nC_3$

Number of triangles that can be formed using n points, out of which m are in the same line $= {}^nC_3 - {}^mC_3$

Number of rectangles in a grid of size $m \times n$ $= \left({}^{m+1}C_2\right)\left({}^{n+1}C_2\right)$

Number of squares in a grid of size $m \times n$ $= m \times n + (m-1) \times (n-1) + (m-2) \times (n-2) + ...$

Number of diagonals in a polygon having n sides

$= \frac{n(n-3)}{2}$

Selection of one or more objects out of p similar objects of one type, q similar objects of one type, and r similar objects of one type is

$(p+1)(q+1)(r+1)-1$

For example, number of ways of selecting at least one book from a bag containing 3 copies of Mathematics books, 2 copies of Physics books and 4 copies of English book is

(3 + 1)(2 + 1)(4 + 1) – 1 = 59

Number of ways of putting *m* distinct objects into *n* different boxes:

Each object has n choices, therefore the number of ways = n^m

Number of non-negative integer solutions of the equation $x_1 + x_2 + ..x_r = n$ = $^{n+r-1}C_{r-1}$

Number of ways in which $(a + b + c + d)$ distinct objects can be divided into 4 groups having a, b, c and d objects = $\frac{(a+b+c+d)!}{(a!)(b!)(c!)(d!)}$

Number of ways in which $4n$ distinct objects can be divided into 4 identical groups = $\frac{(4n)!}{(n!)^4 4!}$

Derangement:

A derangement is a permutation of a set in which no element appears in its original position. For example, if there are n letters and n corresponding envelopes, then number of derangement is the number of ways of putting the letters into the envelopes such that no letter goes to its envelop. It is given by

$D_n = n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-.......+(-1)^n\frac{1}{n!}\right)$, where D_n is the number of derangements. For example $D_1 = 0, D_2 = 1, D_3 = 2, D_4 = 9, D_5 = 44$